Innovamat

The Basic Operations

Everything you do not see about addition, subtraction, multiplication, and division

Foreword

Mathematics goes beyond mere operations. **When learning these fundamental aspects, what matters most?** Is it understanding what we're doing, completing tasks quickly, or achieving accuracy?

The truth is that **all three goals are crucial**. **Understanding** forms the foundation of all learning. It is equally important **to arrive at the correct result** and, through practice, **to develop speed** in calculations—an essential part of **arithmetic fluency**.

Achieving this balance involves **building knowledge** through conversation and guided discovery, and through **practice**, which fosters fluency in operational skills.

In this context, **the teacher's role as a guide is vital**. They help students **discover** strategies in a **clear and transparent** manner, ensuring that **content is consolidated** while encouraging students to step outside their comfort zones.

What do we understand by fluency?

Fluency refers to the ability to work with numbers, operations, and more complex procedures with agility.

It involves not only solving operations quickly but also being **efficient and flexible** in choosing the most appropriate method to tackle a problem, considering the context and the numbers involved.

For this reason, we promote the development of a **wide range of strategies** in the classroom, that ensure the development of this reasoning and flexibility. Each strategy follows a **learning sequence** based on the **CRA model (Concrete, Representational, Abstract)** to ensure mastery and understanding. This process means:

1. Starting with hands-on manipulation using different materials (concrete).

2. Representing what was done manipulatively on paper (representational).

3. Moving on to abstract representations, such as algorithms (abstract).

The significance of learning basic operations

Mathematics, particularly the area of numeration, is a hierarchical discipline, meaning that a solid understanding of one concept is essential before advancing to the next. **Basic operations**—addition, subtraction, multiplication, and division—serve as the foundation for more complex concepts encountered in algebra and calculus.

Beyond just knowing the algorithms for these operations, it is crucial to grasp their meanings and understand how to solve problems involving them.

Each operation can be likened to an iceberg:**what is visible at the surface represents only a small fraction of its overall complexity**. The underlying support, which remains hidden, is what provides clarity and depth to the mastery of each operation.

Thus, we have developed a resource. A booklet that visually summarizes the essential aspects of mastering each basic operation, including:

- The meaning of addition, subtraction, multiplication, and division.

- The proposed methods for solving each operation.

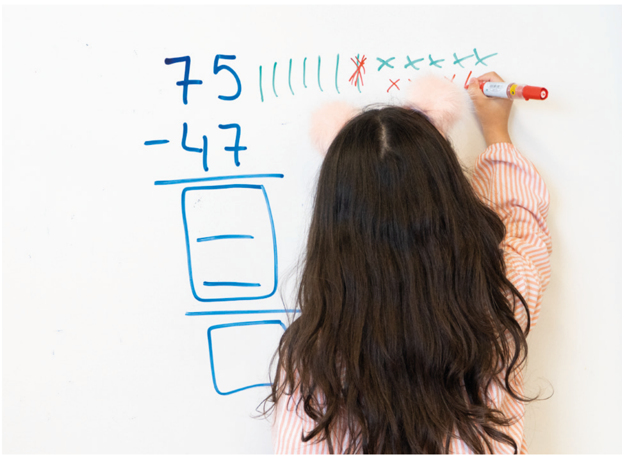
- The learning sequence for each strategy, based on the CRA (Concrete-Representation-Abstract) model.

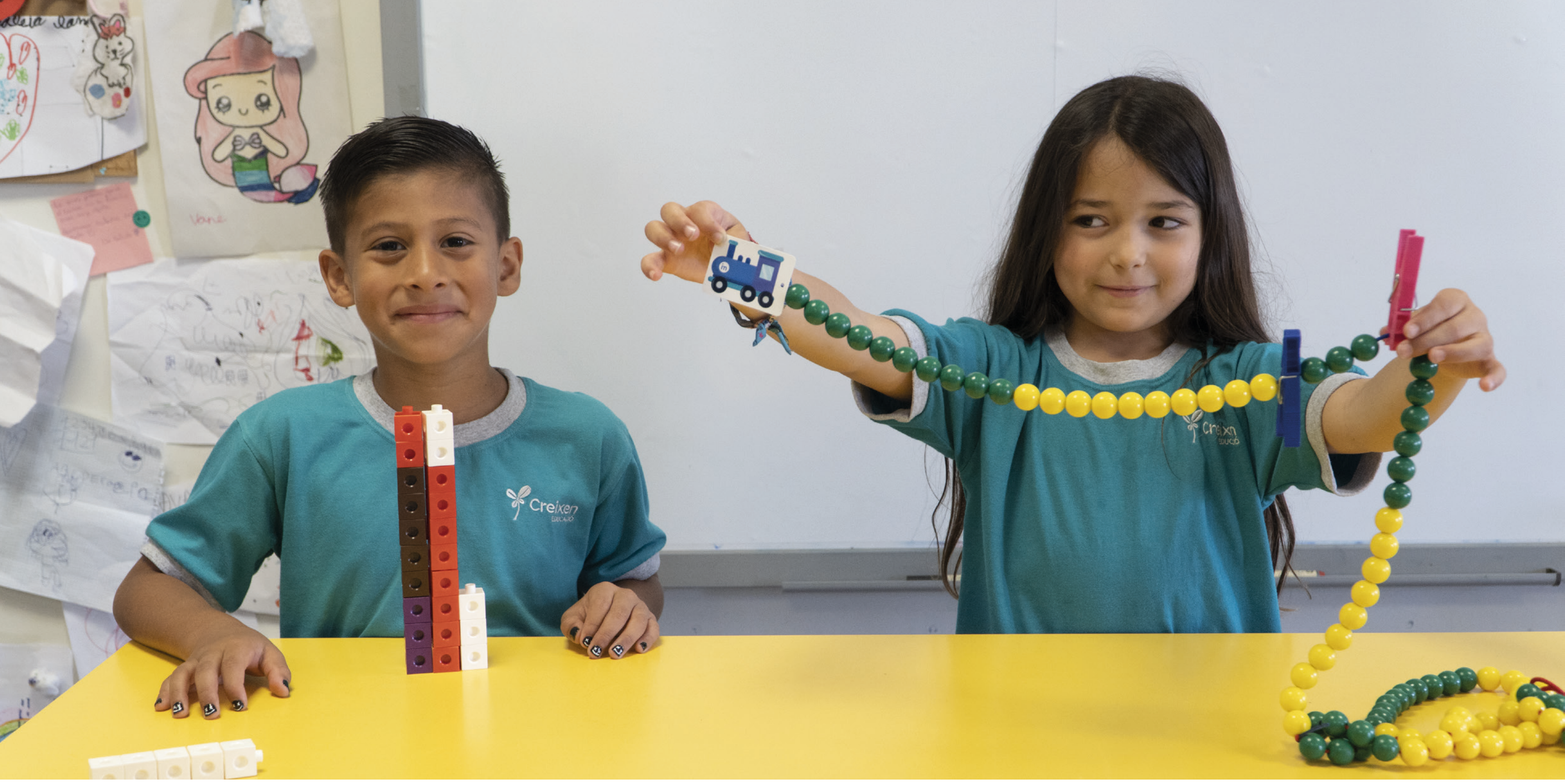
Gaining this knowledge will enable us to be more **flexible and efficient**. However, practice will be crucial for improving calculation speed, a skill we will focus on later.

For now, we encourage you to engage with this resource and use each operation as an opportunity to think critically, understand the concepts, and make progress.

Let's begin!

Scan me
to learn
more





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Division

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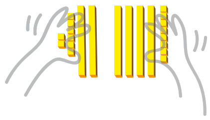
The addition

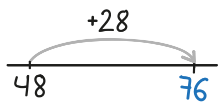
48+28=

UNDERSTAND THE CONCEPT OF ADDITION

Combine



Add



Addition-subtraction relationship

76-28=48

76	
28	48

76-48=28

48+28=76

28+48=76

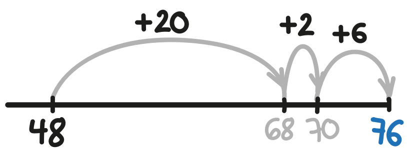
KNOW HOW TO PERFORM THE OPERATION FLUENTLY

EXACT CALCULATION

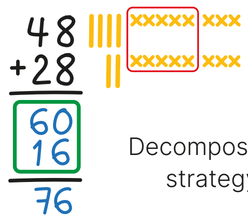
Developing automaticity in single-digit addition

1+1	1+2	1+3	1+4	...
2+1	2+2	2+3	2+4	...
3+1	3+2	3+3	3+4	...
4+1	4+2	4+3	4+4	...
...	...	...	...	...

Jump strategy



Decomposition strategy



Search for equivalences

(Known facts-derived facts)

48+28 =?

↓+2 ↓+2 ↓+4

50+30=80

↓-2 ↓-2 ↓-4

48+28=76

...

ESTIMATIVE CALCULATION

Estimates

48+28 ≃ 50+30

48+28 ≃ 80

45+25 < 48+28 < 50+30

70 < 48+28 < 80

What is addition?

Addition is the first basic operation learned in primary education. It is a very prevalent operation in our daily lives, appearing in many everyday situations. Before starting to solve addition problems, it is important to **understand what it means to add**. For this reason, educators create various situations and contexts in the classroom that help students understand its meaning.

Some situations focus on determining the result of **adding items to an initial quantity**. Others emphasize finding the result of **combining multiple groups of elements**.

How do we solve addition?

Mastering addition goes beyond applying an algorithm. While students will eventually understand the algorithm, **different strategies** are implemented in the classroom to help them develop judgment and flexibility in performing operations.

These strategies include:

- **Jumping along the number line**.

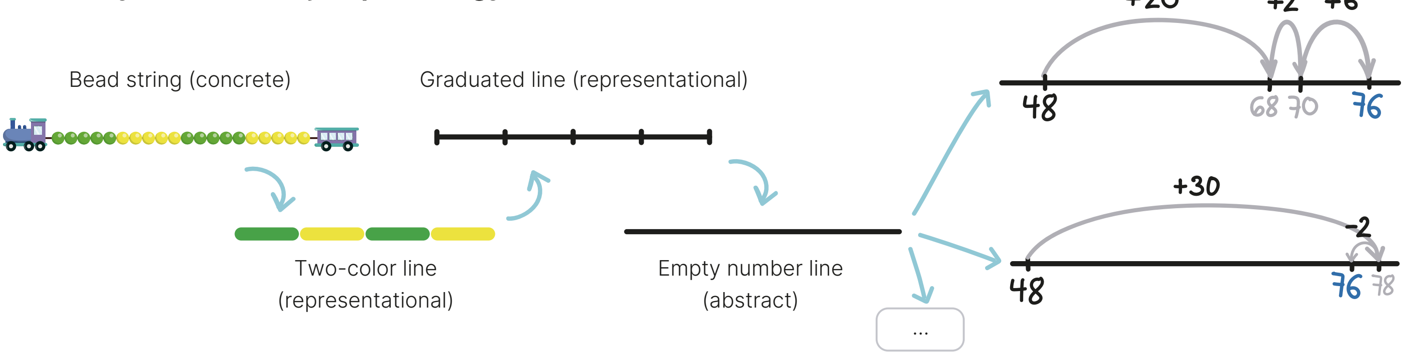
- **Decomposition strategy**.

Developing automaticity in single-digit addition

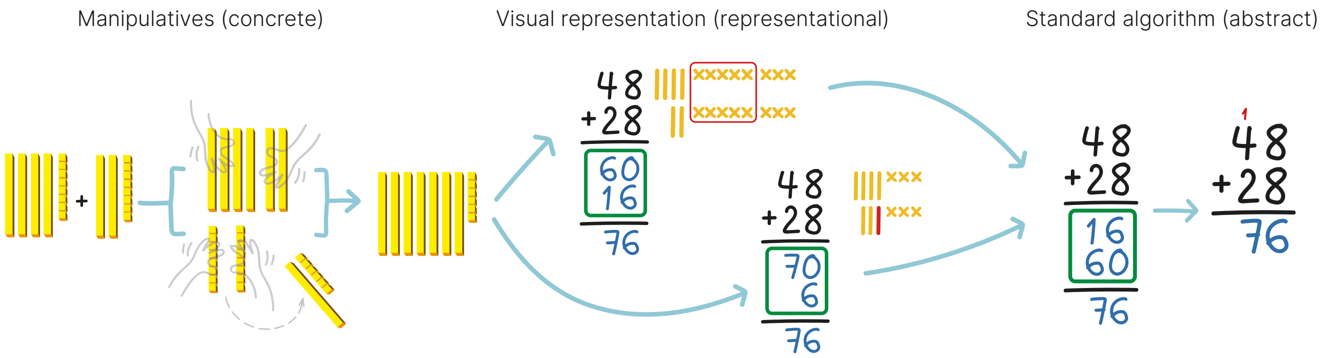
To effectively tackle more complex additions, **it is essential to develop automaticity in single-digit additions**. This means being able to quickly recall results. This allow us to focus on more advanced concepts without getting bogged down by simple calculations.

1+1	1+2	1+3	1+4	...
2+1	2+2	2+3	2+4	...
3+1	3+2	3+3	3+4	...
4+1	4+2	4+3	4+4	...
...	...	...	...	...

How do you build the jump strategy?



How do you build the decomposition strategy?



What the jump strategy enables:

· Enhance mental calculation of sums.
· Solve sums very efficiently.
· Move away from finger counting.

What the decomposition strategy enables:

· Enhance written calculation of sums.
· Understand the position and value of numbers.
· Clearly outline the standard algorithm for addition.

Timeline of a student's learning journey in addition

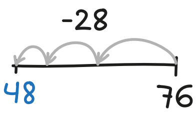
While variations may exist among students, the jump strategy is introduced throughout 1st grade to help students achieve proficiency with the empty number line by the end of the school year. The decomposition strategy is emphasized in 2nd grade, aiming to master the standard algorithm for addition by the end of the year. As the numerical range expands (e.g., from 0 to 10 or from 20 to 50), manipulatives are reintroduced to conduct another cycle of abstraction, progressively moving away from tangible aids.

The Subtraction

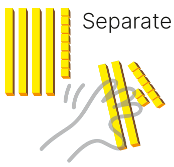
$76 - 28 =$

UNDERSTAND THE CONCEPT OF SUBTRACTION

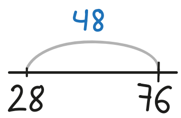
Remove



Separate



Distance



Addition-subtraction relationship

$76-28=48$ | $76-48=28$

76	
28	48

$48+28=76$ | $28+48=76$

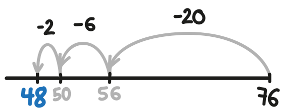
KNOW HOW TO PERFORM THE OPERATION FLUENTLY

EXACT CALCULATION

Developing automaticity in single-digit subtractions

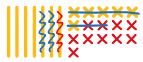
1−1	1−2	1−3	1−4	...
2−1	2−2	2−3	2−4	...
3−1	3−2	3−3	3−4	...
4−1	4−2	4−3	4−4	...
...	...	...	...	...

Jump Strategy



Decomposition strategy

$$\begin{array}{r} \overset{6}{\cancel{7}}\overset{16}{\cancel{6}} \\ -28 \\ \hline 40 \\ 8 \\ \hline 48 \end{array}$$



Search for equivalences
(Known facts-derived facts)

$76-28=?$
$\downarrow +2 \quad \downarrow +2$
$78-30=48$

$76-28=?$
$\downarrow +2 \quad \downarrow +2$
$78-28=50$
$\downarrow -2 \quad \downarrow -2$
$76-28=48$

...

ESTIMATIVE CALCULATION

Estimates

$76-28 \simeq 75-30$
$\downarrow \quad \downarrow$
$76-28 \simeq 45$

$70-30 < 76-28 < 80-20$
$40 < 76-28 < 60$

What is subtraction?

Subtraction is generally viewed as the opposite operation to addition. That is, **the removal of elements from an initial quantity**.

However, subtraction can also be understood as **separating elements from a group** or **finding the distance** between two numbers For example, it answers the question, "What is the length of the jump between 28 and 76?"

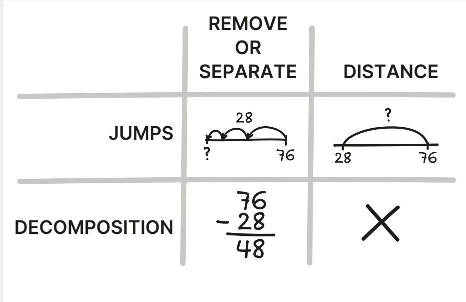
How do we solve subtraction?

There are **various strategies** to solve subtractions. To develop judgment and **flexibility** in calculations, it is essential to know and master each one.

The two main strategies for solving subtractions are:

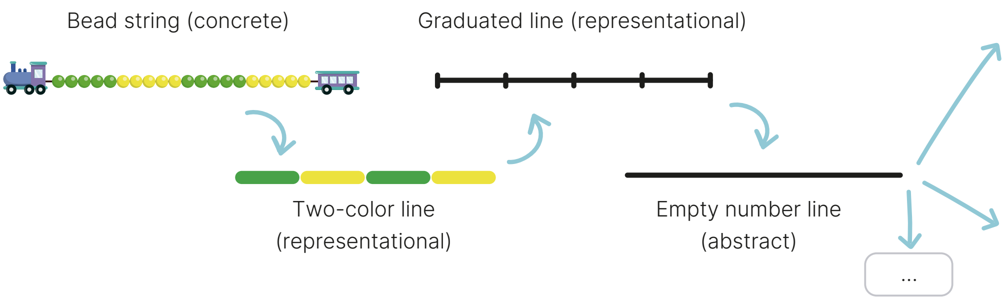
- **Jumps along the number line**: Subtraction is seen as removing elements (jumping backward) or finding the distance between two numbers.

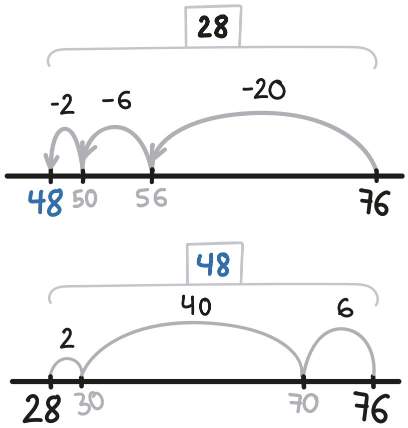
- **Decomposition strategy**: Subtraction is viewed as removing or separating elements from an initial quantity.



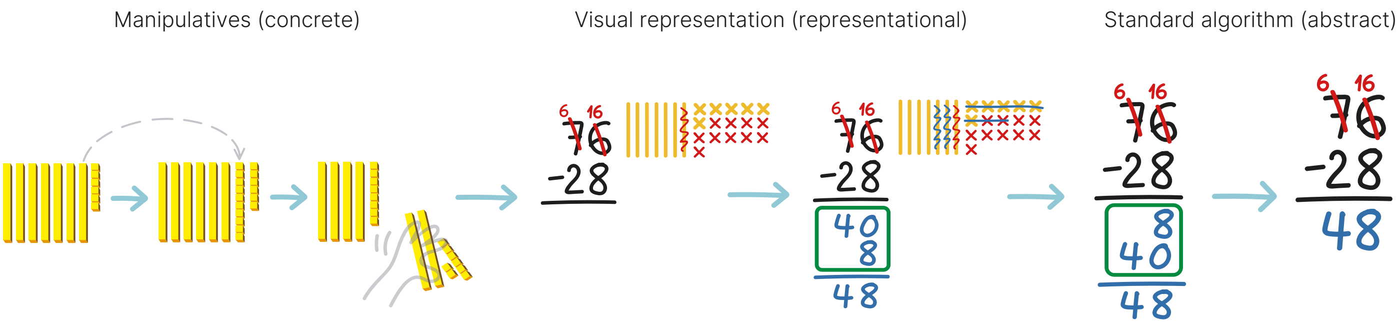
Subtraction is complex in two ways. It can be understood with dual meanings, and it can be solved through two main strategies.

How do you build the jump strategy?





How do you build the decomposition strategy?



What the jump strategy enables:

· Enhance mental calculation of subtractions.
· Solve subtractions very efficiently.
· Move away from finger counting.

What the decomposition strategy enables:

· Enhance written calculation of subtractions.
· Understand the position and value of numbers.
· Clearly outline the standard algorithm for subtraction.

Timeline of a student's learning journey in subtraction

While variations may exist among students, the jump strategy is introduced throughout 1st grade to help students achieve proficiency with the empty number line by the end of the school year. The decomposition strategy is emphasized in 2nd grade, aiming to master the standard algorithm for subtraction by the end of the year. As the numerical range expands (e.g., from 0 to 10 or from 20 to 50), manipulative materials are reintroduced to conduct another cycle of abstraction, progressively moving away from tangible aids.

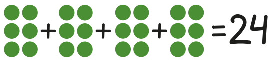
The
multiplication

15×12 =

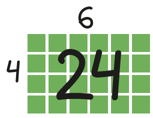
UNDERSTAND THE CONCEPT OF MULTIPLICATION

(4 × 6)

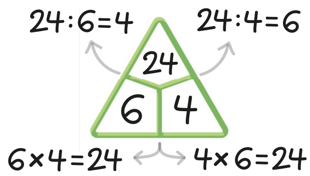
Grouping model



Area model



Relationship: multiplication -division



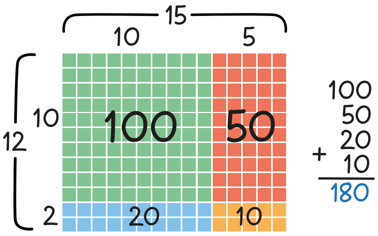
KNOW HOW TO PERFORM THE OPERATION FLUENTLY

EXACT CALCULATION

Developing automaticity in single-digit multiplications (multiplication tables)

1×1	1×2	1×3	1×4	...
2×1	2×2	2×3	2×4	...
3×1	3×2	3×3	3×4	...
4×1	4×2	4×3	4×4	...
...	...	...	...	...

Decomposition Strategy area model



100
50
20
+ 10
180

Equations (Known facts-derived facts)

×2 :2

15×12 = ? 30×6 = 180 → 15×12=180

...

ESTIMATIVE CALCULATION

Estimates

15×12 ≃ 15×10
15×12 ≃ 150

15×10 < 15×12 < 20×12
150 < 15×12 < 240

What is multiplication?

Multiplication is a fundamental operation that involves repeatedly adding the same quantity of elements (iterative addition).

This widely known definition captures only a part of multiplication's essence, as it also involves calculating the **number of elements arranged in the rows and columns of a rectangle**.

How do we solve multiplication?

Various strategies and automatisms exist to master and ensure flexibility in multiplication for children.

Developing automaticity in single-digit multiplications

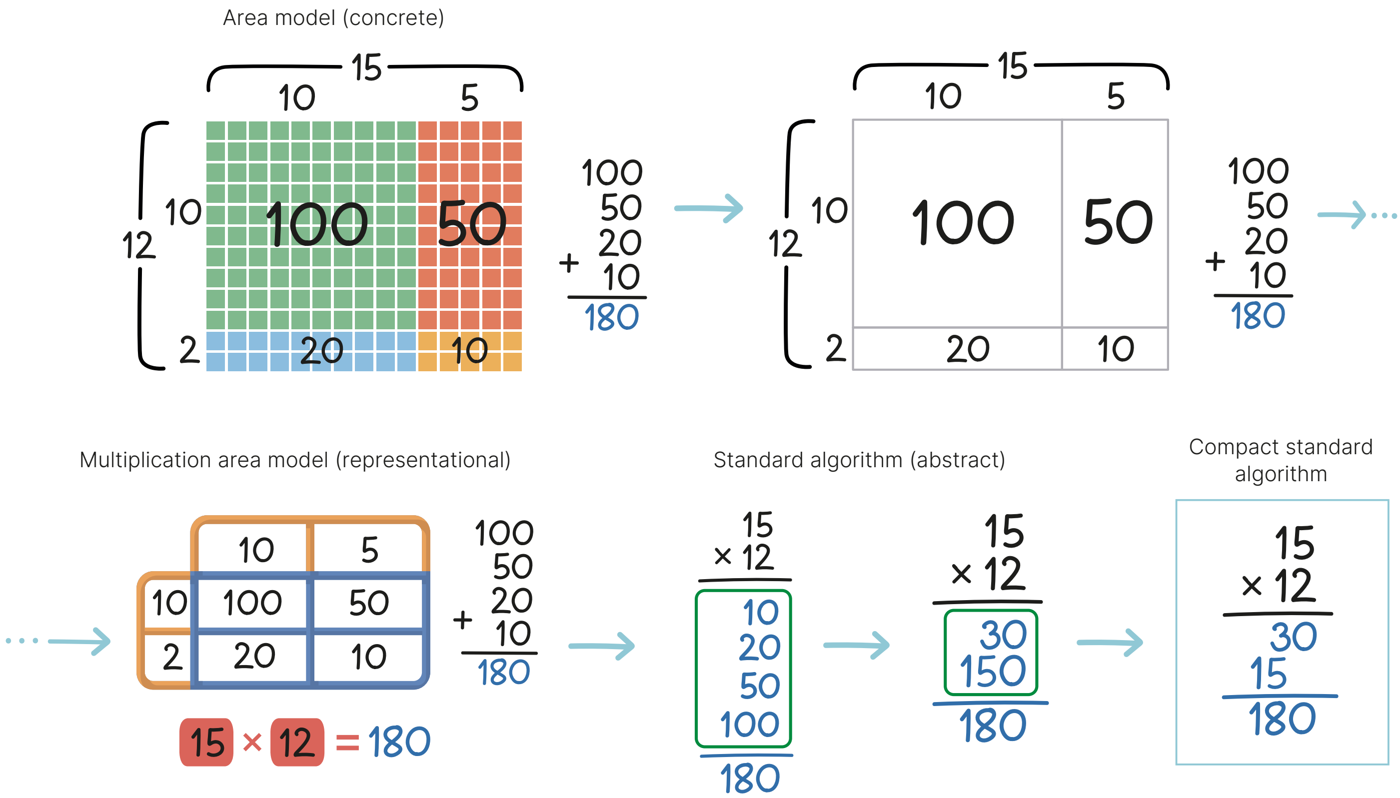
It consists of **quickly recalling** the results of single-digit multiplications (**multiplication tables**). This allows a focus on more advanced or complex concepts without the burden of performing basic calculations.

1×1	1×2	1×3	1×4	...
2×1	2×2	2×3	2×4	...
3×1	3×2	3×3	3×4	...
4×1	4×2	4×3	4×4	...
...	...	...	...	...

The area model strategy

In the classroom, we build **the area mode**, which is **a strategy based on breaking down numbers**. This approach helps us visualize multiplication as the arrangement of elements in a rectangle. Additionally, it provides a clear path to understanding the **standard algorithm for multiplication**.

How do you build the area model?



Timeline of a student's learning journey in multiplication

While variations may exist among students, the multiplication tables are developed throughout 3rd grade until they develop automaticity by the end of the school year. The area model is introduced in 4th grade, aiming for students to master the standard multiplication algorithm by year-end. As operations become more complex, such as multiplying by two-digit numbers, concrete representations are re-examined to enable another cycle of abstraction before being phased out.

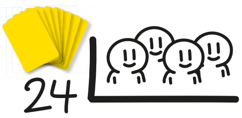
The division

158 : 3 =

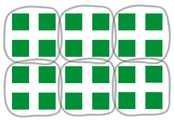
UNDERSTAND THE CONCEPT OF DIVISION

(24 : 4)

Distribute between...

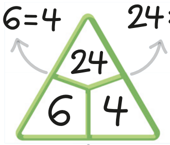


Make groups of...



Relationship: multiplication -division

24:6=4 24:4=6



6×4=24 ↔ 4×6=24

KNOW HOW TO PERFORM THE OPERATION FLUENTLY

EXACT CALCULATION

Strategy by distribution

158	3	
− 30	10	(10×3=30)
128		
− 60	20	(20×3=60)
68		
− 60	20	(20×3=60)
8		
− 6	2	(2×3=6)
2		

158 : 3 = 52 R2

Decomposition strategy

158 = 120 + 30 + 8

120 : 3 = 40
30 : 3 = 10
8 : 3 = 2 R 2

158 : 3 = 40 + 10 + 2 R2
158 : 3 = 52 R2

Search equivalences (Known facts-derived facts)

158 : 3 = ? → 150 : 3 = 50
153 : 3 = 51
156 : 3 = 52 → 158 : 3 = 52 R 2
159 : 3 = 53

(+3 / +1)

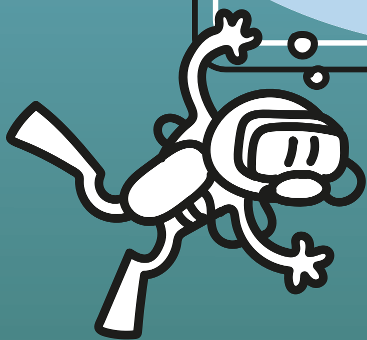
...

ESTIMATIVE CALCULATION

Estimates

158 : 3 ≃ 150 : 3
158 : 3 ≃ 50

150 : 3 < 158 : 3 < 180 : 3
50 < 158 : 3 < 60



What is division?

Distribute or form groups, that is the question. Division is the fourth basic operation taught in elementary school. To convey its meaning, educators present various contexts and situations that illustrate what **it means to divide**.

The most widespread understanding of division is **equitable distribution**, but it also involves **creating equal groups**. For instance, calculating how many groups of 4 beads can be formed from 24 beads.

How do we solve division?

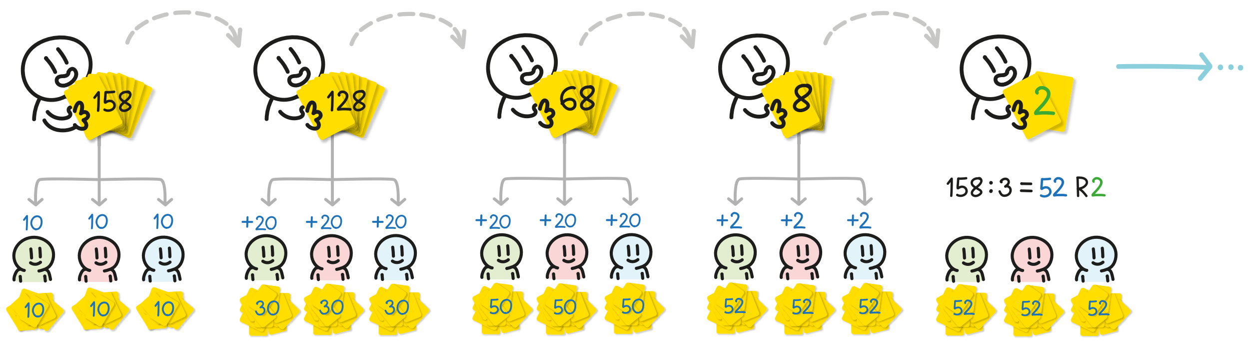
There are several methods for solving divisions. Consequently, **different strategies are developed to foster judgment and flexibility in calculations. Two of the strategies that are taught include:**

-Distribution strategy: This strategy views division as the act of distributing elements. It follows a well-defined learning sequence based on the CRA model, starting from the distribution of materials and progressing to the standard algorithm.

Decomposition strategy: This strategy is particularly useful for solving divisions mentally. It does not have as structured a learning sequence but lays the foundation for working with equivalences.

How do you build the distribution strategy?

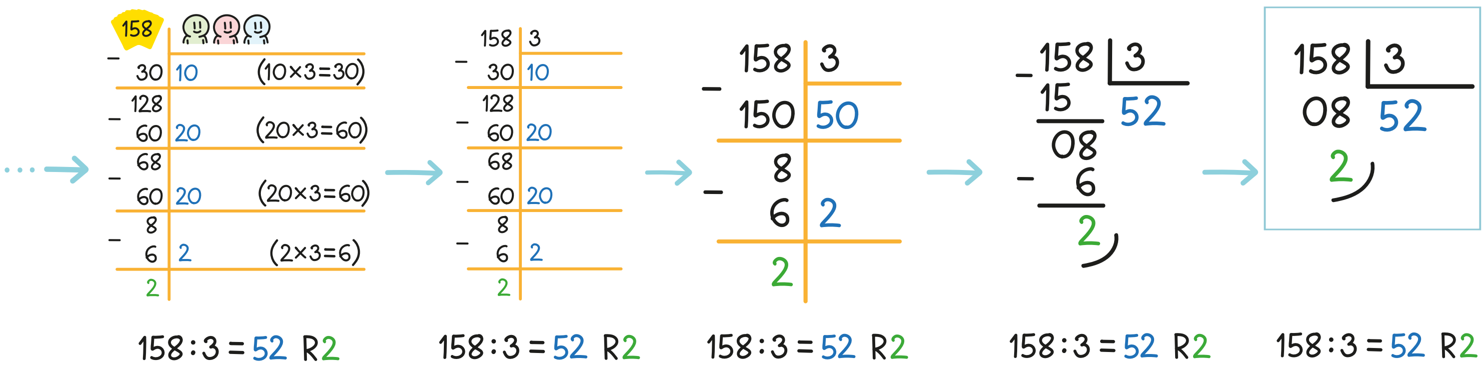
Distribute elements (concrete)



Repeated subtraction (representational)

Standard algorithm (abstract)

Compact standard algorithm



Timeline of a student's learning journey in division during elementary school

While there may be variations among students, the construction and consolidation of the two division strategies are typically achieved by the end of fourth grade, as they begin to master the standard division algorithm. As operations become more complex, such as dividing by two-digit numbers, concrete representations are revisited. This allows for another cycle of abstraction before gradually phasing them out.

Timeline of a student's learning journey in strategies throughout the grades

	ADDITION	SUBTRACTION	MULTIPLICATION	DIVISION
1st	Understanding early addition with the jump strategy: From bead strings to the empty number line within the range of 0-100 Beginning to develop automaticity in single-digit additions. The use of addition-subtraction relationship with addition squares.	Understanding early subtraction with the jump strategy: From bead strings to the empty number line within the range of 0-100 Beginning to develop automaticity of single-digit subtractions. The use of addition-subtraction relationship with addition squares.		
2nd	Understanding jumps along the number line with ease. Understanding the decomposition strategy up to the standard addition algorithm in the range 0-100. Developing automaticity in single-digit addition.	Understanding jumps along the number line with ease. Understanding the decomposition strategy up to the standard subtraction algorithm in the range 0-100. Developing automaticity in subtraction single-digit addition.	Early concepts of multiplicative thinking: Doubles and halves	
3rd	Understanding of addition using both strategies within the range of 0-10,000. Fluency with strategies involving numbers up to two digits.	Understanding of subtraction using both strategies within the range of 0-10,000. Fluency with strategies involving numbers up to two digits.	Begin developing automaticity in multiplication tables. Understanding the area model up to the multiplication diagram.	Use of the multiplication-division relationship with multiplication triangles.
4th	Fluency in addition up to 4 digits	Fluency in subtraction up to 4 digits	Understanding the standard algorithm for multiplication.	Understanding the distribution strategy up to distribution optimization, as done with the standard algorithm, and the decomposition strategy. for division.
5th 6th	Understanding and fluency in the addition of natural numbers through the jump strategy and the decomposition strategy. Understanding addition with decimal numbers.	Understanding and fluency in the subtraction of natural numbers through the jump strategy and the decomposition strategy. Understanding subtraction with decimal numbers.	Consolidation of the standard algorithm for multiplication over larger ranges. Multiplication fluency with natural numbers. Understanding multiplication with decimal numbers.	Consolidation of distributions with the minimal number of distributions, as done with the standard algorithm. Multiplication fluency with natural numbers. Understanding division with decimal numbers.