Innovamat

Understanding the Basic Operations

Unveiling the Essentials of Addition, Subtraction, Multiplication, and Division

Foreword

Mathematics goes beyond mere operations. **When learning these fundamental skills, what matters most—** understanding, speed, or accuracy?

The truth is that **all three goals are crucial**. **Understanding** forms the foundation of all learning. It is equally important **to arrive at the correct result** and, through practice, **to develop speed** in key components of **arithmetic fluency**.

Achieving this balance requires **building knowledge** through conversation, guided discovery and **practice**, fostering fluency in operational skills.

In this process, **the teacher's role as a guide is vital**. They help students **discover** strategies in a **clear and transparent** manner, ensuring that **content is consolidated** while encouraging students to step outside their comfort zones.

What do we understand by fluency?

Fluency refers to the ability to work with numbers, operations, and more complex procedures with agility.

It encompasses solving problems **efficiently and flexibly**, selecting the best approach based on the context and numbers involved.

For this reason, we promote the development of a **wide range of strategies** in the classroom, that ensure the development of this reasoning and flexibility. Each strategy follows a **learning sequence** based on the **CRA model (Concrete, Representational, Abstract)** to ensure mastery and deepen understanding. This process means:

1. Starting with hands-on manipulation using different materials (concrete).

2. Representing what was done manipulatively on paper (representational).

3. Moving on to abstract representations, such as algorithms (abstract).

The significance of learning basic operations

Mathematics, particularly the area of numeration, is a hierarchical discipline, meaning that a solid understanding of one concept is essential before advancing to the next. **Basic operations**—addition, subtraction, multiplication, and division—are the foundation for advanced mathematical concepts, such as algebra and calculus..

Beyond just knowing the algorithms for these operations, it is crucial to grasp their meanings and understand how to solve problems involving them.

Each operation is like an iceberg: **what you see on the surface is only a small part of its overall complexity**. The hidden foundations provide clarity and depth, enabling true mastery of each operation.

Thus, we have developed a resource. A booklet that visually summarizes the essential aspects of mastering each basic operation, including:

- The meaning of addition, subtraction, multiplication, and division.

- The proposed methods for solving each operation.

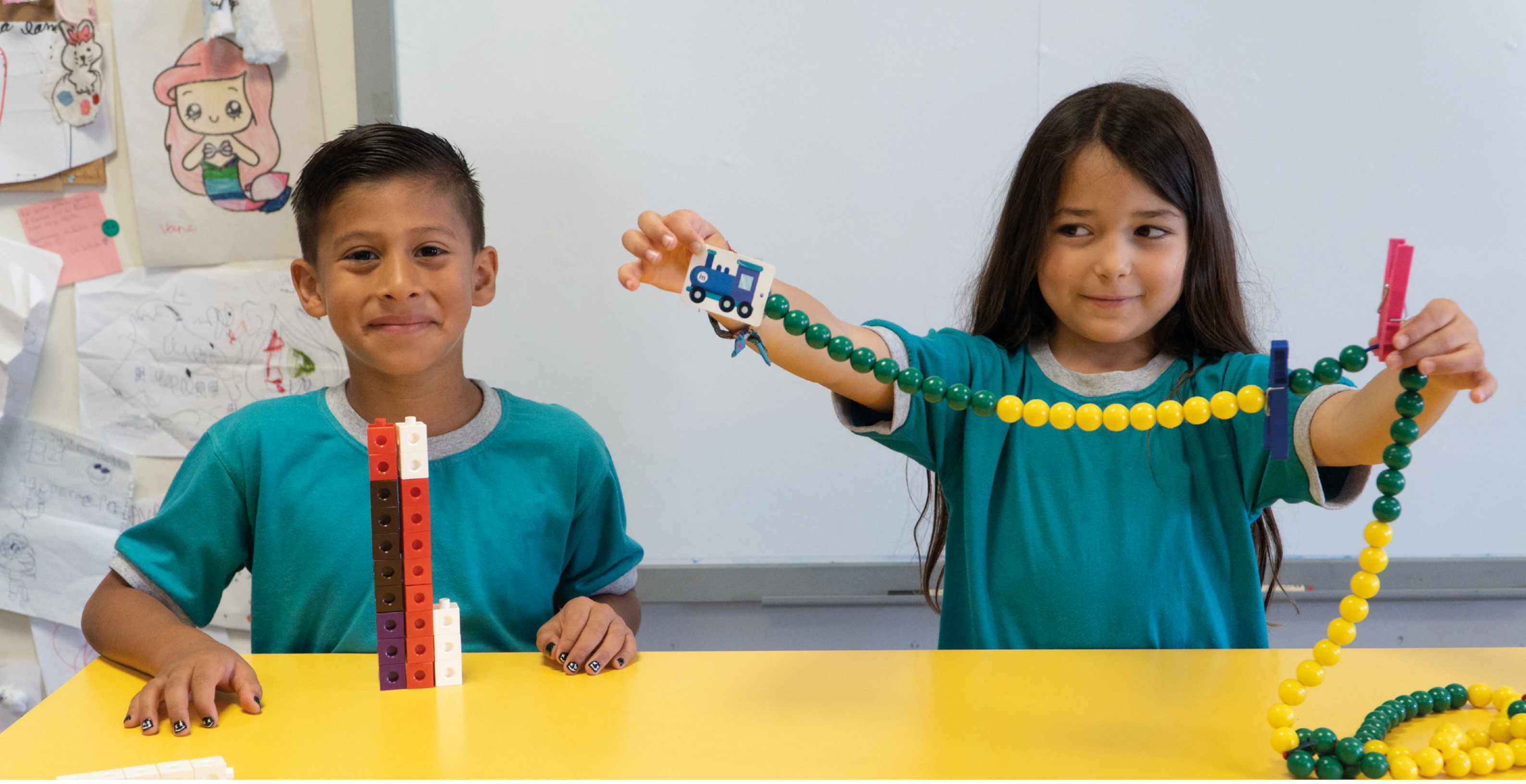
- The learning sequence for each strategy, based on the CRA (Concrete-Representation-Abstract) model.

This knowledge will enable greater **flexibility and efficiency** in problem-solving. Practice, however, is key to improving calculation speed—a skill we will explore in depth later.

For now, we encourage you to engage with this resource and use each operation as an opportunity to think critically, understand the concepts, and make progress.

Let the journey begin!

Scan me to learn more



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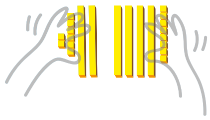
The addition

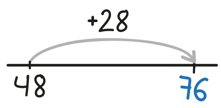
$48+28=$

UNDERSTAND THE CONCEPT OF ADDITION

Combine



Add



Addition-subtraction relationship

$76-28=48$ | $76-48=28$

76	
28	48

$48+28=76$ | $28+48=76$

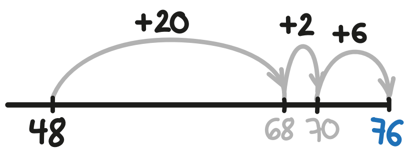
DEVELOP FLUENCY

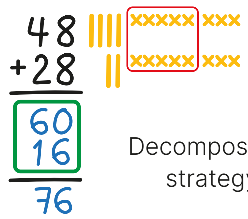
EXACT CALCULATION

Develop fluency
in single-digit addition

1+1	1+2	1+3	1+4	...
2+1	2+2	2+3	2+4	...
3+1	3+2	3+3	3+4	...
4+1	4+2	4+3	4+4	...
...	...	...	...	...

Jump strategy





Decomposition
strategy

Identify equivalences
(Known facts-
derived facts)

$48+28=?$
(+2, +2, +4)
$50+30=80$
(−2, −2, −4)
$48+28=76$

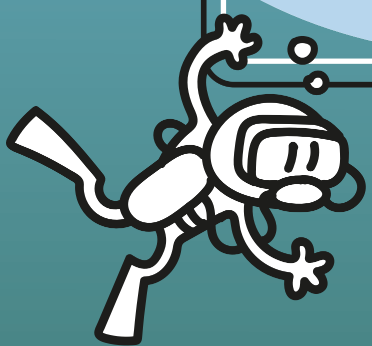
...

ESTIMATION CALCULATION

Estimation

$48+28 \simeq 50+30$
$48+28 \simeq 80$

$45+25 < 48+28 < 50+30$
$70 < 48+28 < 80$



What is addition?

Addition is the first basic operation introduced in primary education. It is a fundamental concept, frequently encountered in everyday life. Before starting to solve addition problems, it is important to **understand what it means to add**. Educators, therefore, create diverse classroom situations and contexts to help convey its meaning.

Some situations focus on determining the total of **adding items to an initial quantity**. Others emphasize **combining multiple groups to find the total**.

How do we solve addition?

Mastering addition goes beyond applying an algorithm. While students will eventually understand the algorithm, **different strategies** are introduced in the classroom to develop judgment and flexibility when performing operations..

These strategies include:

- **Jump strategy**.

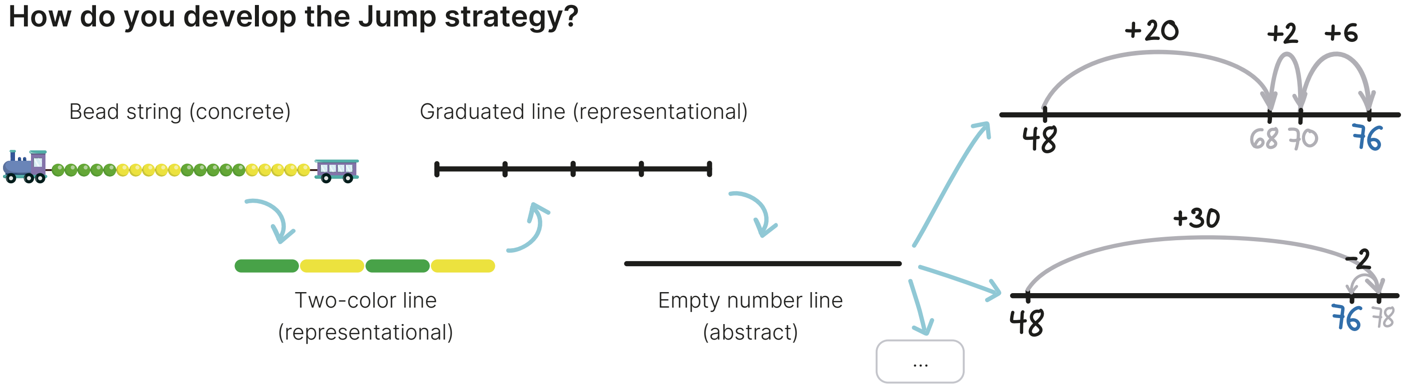
- **Decomposition strategy**.

Developing fluency in single-digit addition

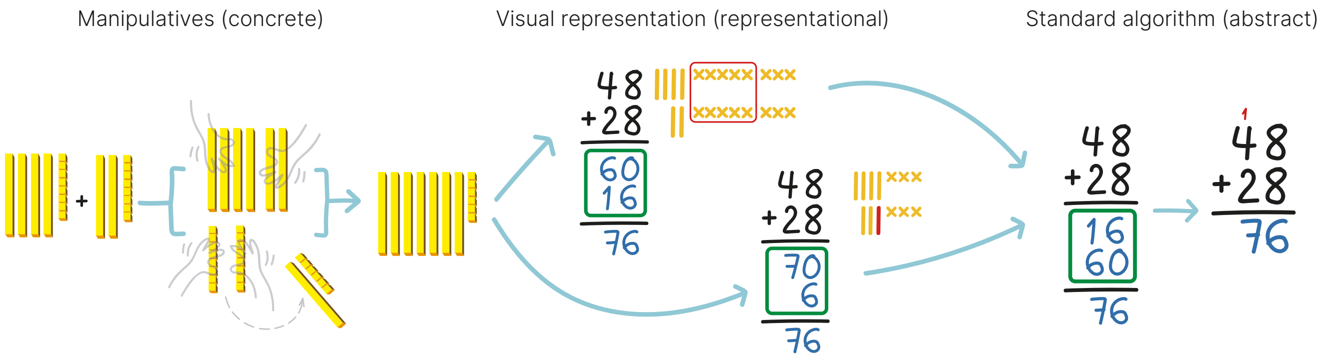
To effectively tackle more complex additions, **it is essential to develop fluency in single-digit additions**. This means being able to quickly recall results. This allow us to focus on more advanced concepts without getting slowed down by simple calculations.

1+1	1+2	1+3	1+4	...
2+1	2+2	2+3	2+4	...
3+1	3+2	3+3	3+4	...
4+1	4+2	4+3	4+4	...
...	...	...	...	...

How do you develop the Jump strategy?



How do you develop the Decomposition strategy?



What the Jump strategy helps to achieve:

· Enhance mental calculation of sums.
· Solve sums very efficiently.
· Move away from finger counting.

What the Decomposition strategy helps to achieve:

· Enhance written calculation of sums.
· Understand the position and value of numbers.
· Clearly outline the standard algorithm for addition.

Timeline of a student's learning journey in addition

While learning progress may vary, the jump strategy is introduced in 1st grade to help students achieve proficiency with the empty number line by the end of the school year. The decomposition strategy becomes a focus in 2nd grade. As the numerical range expands (e.g., from 0 to 10 or from 20 to 50), manipulatives are reintroduced to revisit the abstraction process, progressively moving away from tangible aids.

The Subtraction

$76 - 28 =$

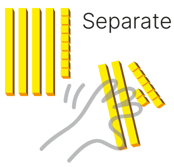
UNDERSTAND THE CONCEPT OF SUBTRACTION

Remove

-28

48 76

Separate



Distance

48

28 76

Addition-subtraction relationship

76-28=48

76-48=28

48+28=76

28+48=76

76	
28	48

DEVELOP FLUENCY

EXACT CALCULATION

Develop fluency in single-digit subtractions

1−1	1−2	1−3	1−4	...
2−1	2−2	2−3	2−4	...
3−1	3−2	3−3	3−4	...
4−1	4−2	4−3	4−4	...
...	...	...	...	...

Jump strategy

-2 -6 -20

48 50 56 76

Decomposition strategy

6 16

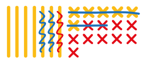
76

-28

40

8

48



Identify equivalences (Known facts-derived facts)

76-28 = ?

+2 +2

78-30=48

76-28 = ?

+2 +2

78-28 = 50

-2 -2

76-28 = 48

...

ESTIMATION CALCULATION

Estimation

76-28 ≃ 75-30

76-28 ≃ 45

70-30 < 76-28 < 80-20

40 < 76-28 < 60

What is subtraction?

Subtraction is generally viewed as the inverse operation of addition. That is, **the removal of elements from an initial quantity**.

However, subtraction can also be understood as **separating elements from a group** or **determining the distance** between two numbers For example, it answers the question, "What is the length of the jump between 28 and 76?"

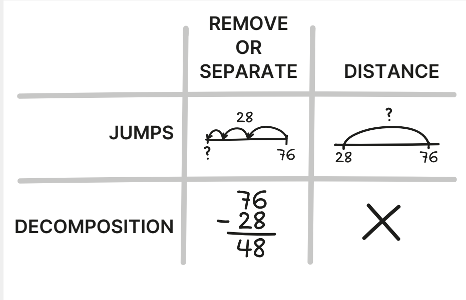
How do we solve subtraction?

There are **different strategies** to solve subtractions. Developing judgment and **flexibility** in calculations, it is essential to know and master each one.

The two main strategies for solving subtractions are:

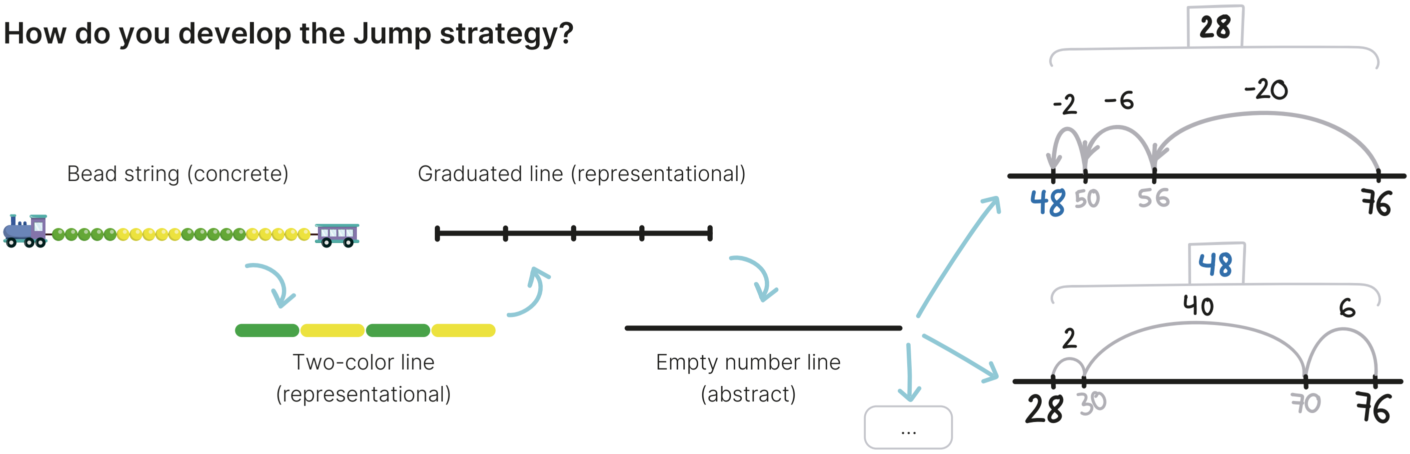
- **Jump strategy**: Subtraction is seen as removing elements (jumping backward) or finding the distance between two numbers.

- **Decomposition strategy**: Subtraction is viewed as removing or separating elements from an initial quantity.

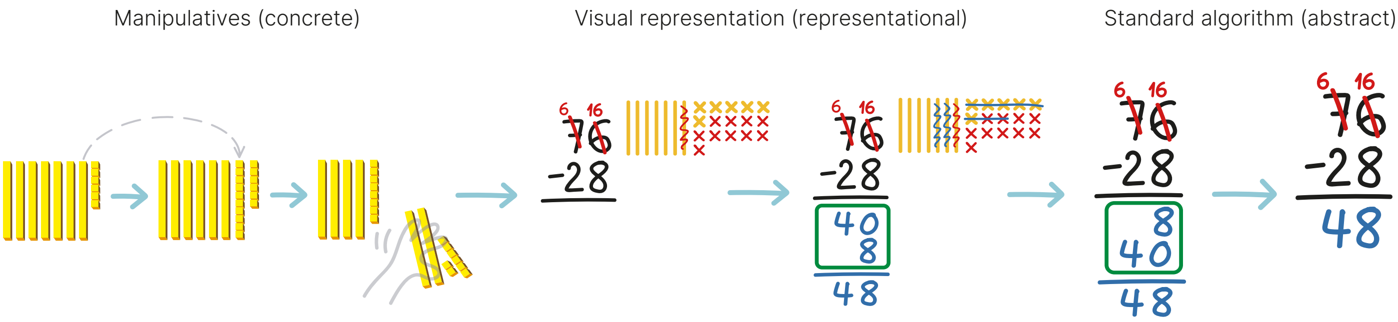


Subtraction is complex for two reasons: it involves dual meanings and can be solved using two main strategies.

How do you develop the Jump strategy?



How do you develop the Decomposition strategy?



What the Jump strategy helps to achieve:

· Enhances mental subtraction skills.
· Promotes efficient subtraction problem-solving.
· Encourages moving beyond finger counting.

What the Decomposition strategy helps to achieve:

· Enhances written subtraction skills.
· Deepens understanding of place value and number positioning.
· Provides a clear foundation for the standard subtraction algorithm.

Timeline of a student’s learning journey in subtraction

While variations may exist among students, the jump strategy is introduced throughout 1st grade to help students achieve proficiency with the empty number line by the end of the school year. The decomposition strategy is emphasized in 2nd grade. As the numerical range expands (e.g., from 0 to 10 or from 20 to 50), manipulative materials are reintroduced to conduct another cycle of abstraction, progressively moving away from tangible aids.

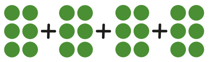
The
multiplication

$$15 \times 12 =$$

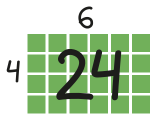
UNDERSTAND THE CONCEPT OF MULTIPLICATION

(4 × 6)

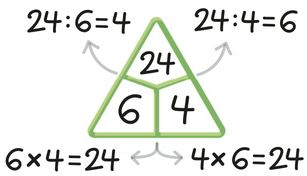
Grouping model



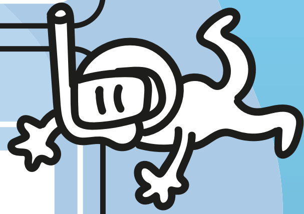
Area model



Relationship:
multiplication
-division



DEVELOP FLUENCY

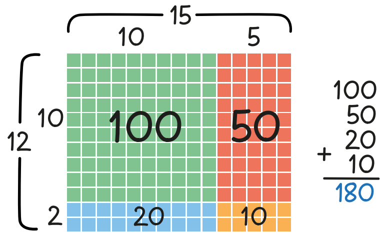


EXACT CALCULATION

Develop fluency in
single-digit multiplications
(multiplication tables)

1×1	1×2	1×3	1×4	...
2×1	2×2	2×3	2×4	...
3×1	3×2	3×3	3×4	...
4×1	4×2	4×3	4×4	...
...	...	...	...	...

Decomposition
strategy
area model



Identify
equivalences
(Known facts-
derived facts)

×2 :2

15×12 = ? 30×6 = 180 → 15×12 = 180

...

ESTIMATION CALCULATION

Estimation

15×12 ≃ 15×10
15×12 ≃ 150

15×10 < 15×12 < 20×12
150 < 15×12 < 240

What is multiplication?

Multiplication is a fundamental operation that represents the repeated addition of the same quantity.

While this definition is well-known, it only partially captures multiplication's essence. Multiplication also includes determining the **number of elements arranged in the rows and columns of a rectangle**.

How do we solve multiplication?

A variety of strategies and techniques are used to help children master multiplication and develop flexibility in its application.

Developing fluency in single-digit multiplications

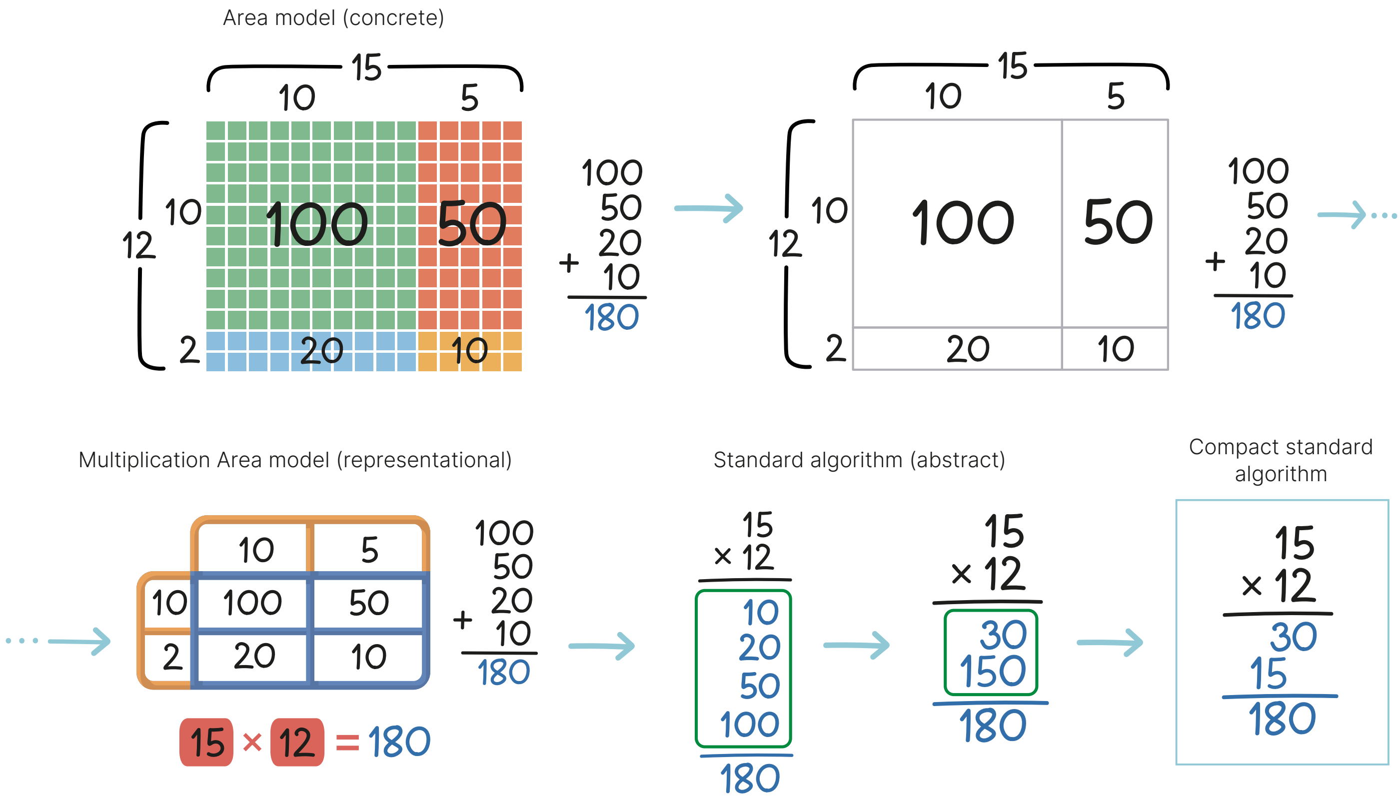
It consists of **quickly recalling** the results of single-digit multiplications (**multiplication tables**). This enables students to focus on more advanced concepts without being slowed by basic calculations.

1×1	1×2	1×3	1×4	...
2×1	2×2	2×3	2×4	...
3×1	3×2	3×3	3×4	...
4×1	4×2	4×3	4×4	...
...	...	...	...	...

The area model strategy

In the classroom, we build **the area model**, which is **a strategy based on breaking down numbers (decomposition)**. This approach helps students visualize multiplication as the arrangement of elements in a rectangle. Additionally, it provides a clear foundation to understanding the **standard algorithm for multiplication**.

How do you develop the area model?



Timeline of a student's learning journey in multiplication

While learning progress may vary among students, the multiplication tables are developed throughout 3rd grade until they develop fluency by the end of the school year. The area model is also introduced in 3rd grade, aiming for students to master the standard multiplication algorithm by year-end. As multiplication problems become more complex, such as multiplying by two-digit numbers, concrete representations are re-examined to support another cycle of abstraction before being phased out.

The division

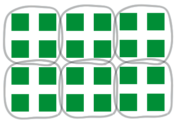
158:3=

UNDERSTAND THE CONCEPT OF DIVISION

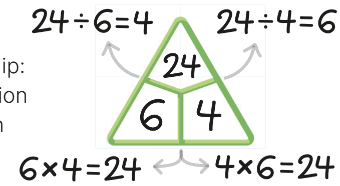
(24 : 4)

Distribute between...

Make groups of...



Relationship: multiplication -division



DEVELOP FLUENCY

EXACT CALCULATION

Distribution Strategy

10+20+20+2

3 | 158

(10×3=30) − 30

128

(20×3=60) − 60

68

(20×3=60) − 60

8

(2×3=6) − 6

2

158÷3 = 52 R2

Decomposition strategy

158 = 120+30+8

120÷3 = 40

30÷3 = 10

8÷3 = 2 R2

158÷3 = 40 + 10 + 2 R2

158÷3 = 52 R2

Identifying equivalences

(Known facts-derived facts)

158÷3 =? → 150÷3 = 50

↓+3 ↓+1

153÷3 = 51

↓+3 ↓+1

156÷3 = 52

↓+3 ↓+1

159÷3 = 53

158÷3 = 52 R2

...

ESTIMATION CALCULATION

Estimation

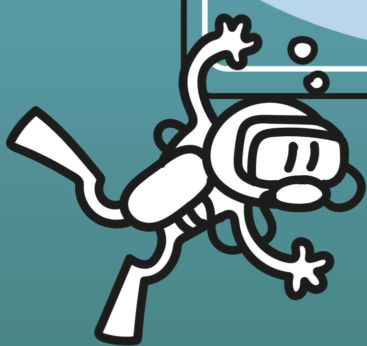
158÷3 ≃ 150÷3

↓ ↓

158÷3 ≃ 50

150÷3 < 158÷3 < 180÷3

50 < 158÷3 < 60



What is division?

To distribute or to form groups—that is the question. Division is the fourth fundamental operation introduced in elementary school. To convey its meaning, educators use a variety of contexts and situations to illustrate **the concept of division**.

The most widespread understanding of division is **equitable distribution**, but it also involves **forming equal groups**. For instance, determining how many groups of 4 beads can be formed from 24 beads.

How do we solve division?

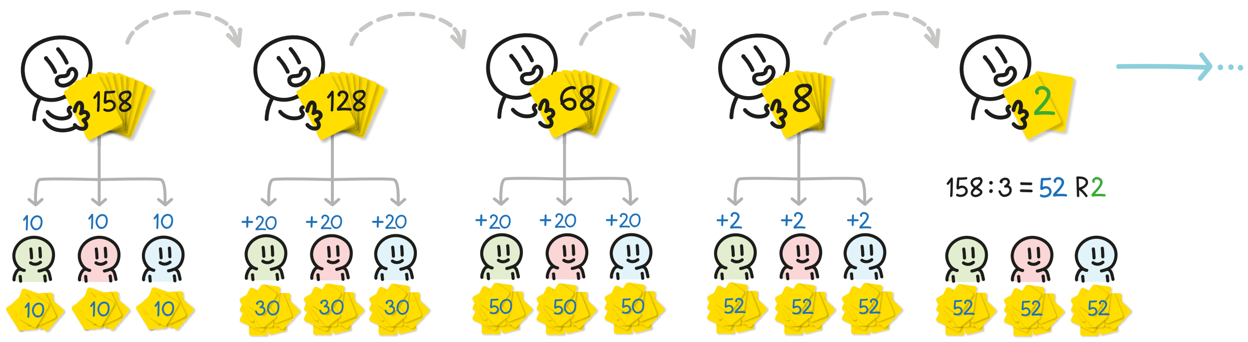
There are several methods for solving division problems. Consequently, **different strategies** are developed to foster **judgment and flexibility in calculations.** Two of the strategies that are taught include:

- **Sharing strategy:** This strategy views division as the act of distributing elements. It follows a structured learning sequence based on the **CRA model**, starting from the physical distribution of materials and progressing to the standard algorithm.

- **Decomposition strategy:** This strategy is particularly useful for solving division problems **mentally**. It does not have as structured a learning sequence but provides a foundation for working with **equivalences**.

How do you develop the Sharing strategy?

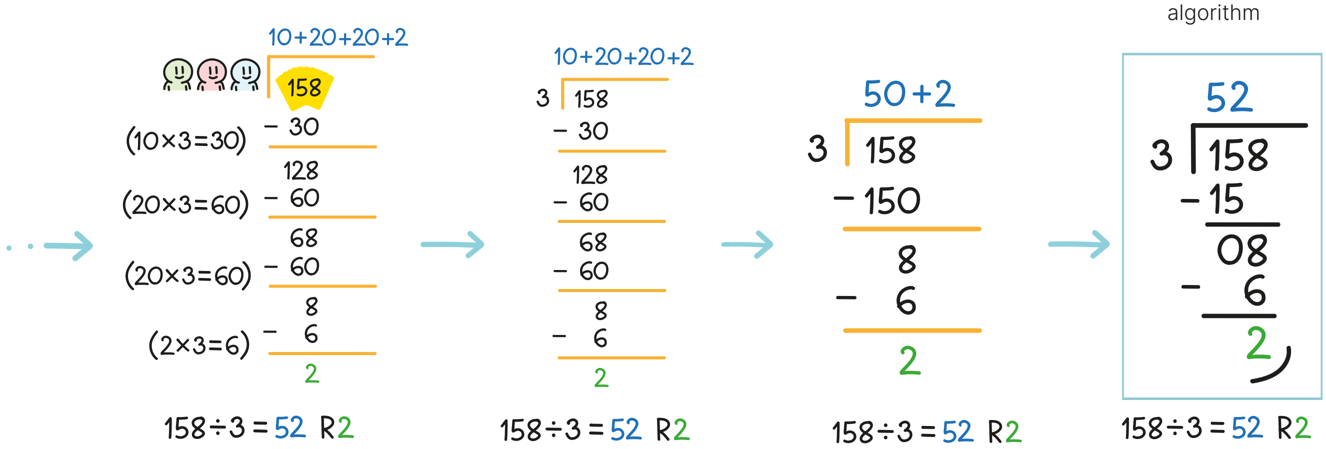
Distribute elements (concrete)



Repeated subtraction (representational)

Standard algorithm (abstract)

Compact standard algorithm



Timeline of a student's learning journey in division during elementary school

While learning progress may vary, the construction and consolidation of the two division strategies are typically achieved by the end of fourth grade, as students begin to master the standard division algorithm. As operations become more complex, such as dividing by two-digit numbers, concrete representations are revisited. This supports another cycle of abstraction before gradually phasing out concrete aids.

Timeline of a student's learning journey in mathematical strategies across the grades

	ADDITION	SUBTRACTION	MULTIPLICATION	DIVISION
1st	Understanding early addition with the jump strategy: From bead strings to the empty number line within the range of 0-100 Automaticity in single-digit additions. Using the addition-subtraction relationship with addition squares.	Understanding early subtraction with the jump strategy: From bead strings to the empty number line within the range of 0-100 Automaticity in single-digit subtractions. Using the of addition-subtraction relationship with addition squares.		
2nd	Understanding jumps along the number line with ease. Understanding the decomposition strategy in the range 0-1000. Fluency in addition in the range 0-100.	Understanding jumps along the number line with ease. Understanding the decomposition strategy in the range 0-1000. Fluency in subtraction in the range 0-100.		
3rd	Understanding of addition using both strategies within the range of 0-10,000. Fluency with strategies involving numbers up to three digits.	Understanding of subtraction using both strategies within the range of 0-10,000. Fluency with strategies involving numbers up to three digits.	Automaticity in multiplication tables. Understanding the area model.	Use of the multiplication-division relationship with multiplication triangles.
4th	Fluently add multi-digit whole numbers using the standard algorithm.	Fluently subtract multi-digit whole numbers using the standard algorithm.	Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations.	Understanding the distribution strategy leading to distribution optimization, as done with the standard algorithm, and the decomposition strategy. for division.
5th	Understanding and fluency in the addition of natural numbers through the jump strategy and the decomposition strategy.	Understanding and fluency in the subtraction of natural numbers through the jump strategy and the decomposition strategy.	Consolidation of the standard algorithm for multiplication involving multi-digit whole numbers. Multiplication fluency with natural numbers.	Consolidation of distributions with the minimal number of distributions, as done with the standard algorithm. Multiplication fluency with natural numbers.